Heat Generation and Joule Heating Effects on MHD Natural Convection Flow along a Vertical Flat Plate with Viscous Dissipation and Non-Uniform Surface Temperature

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ABSTRACT: *Heat generation and Joule heating synergistically affect magnetohydrodynamic (MHD) natural convection flow along a vertical flat plate with viscous dissipation and a non-uniform surface temperature profile. We use appropriate transformations to turn the governing equations into dimensionless non-similar forms and solve them numerically using the Keller technique. Heat generation, magnetic field intensity, Joule heating, Prandtl number, and viscous dissipation affect velocity, temperature, skin friction, and local heat transmission, as the research examines. The thermal and flow patterns are noteworthy. Heat generation parameter Q intensifies fluid motion, increasing velocity profiles. With increasing Q, temperature profiles rise slightly. As M increases, velocity profiles decrease, while temperature profiles increase. Joule heating described by parameter J, increases velocity and temperature profiles, improving fluid motion and thermal distribution. Additionally, higher Prandtl numbers decrease fluid velocity and boundary layer temperature profiles. As viscous dissipation parameter, Vd, increases, velocity and temperature profiles decrease. The study shows how the parameters affect skin friction and local heat transfer coefficients. These findings affect many engineering applications, including heat transfer and fluid movement in magnetic field systems, providing useful guidance for optimizing performance.*

Keywords: *Heat Generation, Magnetohydrodynamic, Viscous Dissipation, Natural Convection, Joule Heating.*

I. Introduction

The study of heat generation and Joule heating effects on magnetohydrodynamic (MHD) natural convection flow along a vertical flat plate with viscous dissipation and non-uniform surface temperature has been a topic of significant interest in fluid mechanics and heat transfer. Several researchers have explored related phenomena in various contexts, and this literature review summarizes their contributions.

Ackroyd (1962) investigated stress work effects in laminar flat plate natural convection and highlighted the importance of understanding the impact of stress work in such flows. Abo-Eldahab and El Aziz (2005) considered the effects of viscous dissipation and Joule heating in MHD-free convection from a vertical plate with power-law temperature variations. Their work incorporated hall and ion-slip currents, expanding the scope of understanding in MHD flows.

Alam et al. (2007) studied free convection from a vertical permeable circular cone, considering pressure work and non-uniform surface temperature. They explored the implications of these factors on heat transfer. Another study by the same authors (2007) focused on stress work in natural convection flow along a vertical flat plate with Joule heating and heat conduction.

Chowdhury and Islam (2000) delved into MHD free convection flow of visco-elastic fluid past an infinite porous plate, contributing to our understanding of viscoelastic flows in the presence of magnetic fields. El-Amin and Mohammadein (2005) examined the effects of viscous dissipation and Joule heating on magnetohydrodynamic Hiemenz flow of a micropolar fluid, shedding light on the behavior of complex fluids in magnetic fields.

Hossain and Ahmed (1990) focused on MHD forced and free convection boundary layer flow near the leading edge, offering insights into the effects of magnetohydrodynamics on boundary layer flows. Hossain and Paul (2001) investigated free convection from a vertical permeable circular cone with non-uniform surface heat flux, emphasizing the role of surface heat flux variations.

Joshi and Gebhart (1981) explored the effect of pressure stress work and viscous dissipation in natural convection flows, highlighting the significance of pressure work in such scenarios. Kuiken (1970) contributed to the understanding of magneto-hydrodynamic free convection in strong cross-flow fields.

Mahajan and Gebhart (1989) focused on viscous dissipation effects in buoyancy-induced flows, providing valuable insights into flows driven by temperature differences. Mendez and Trevino (2000) studied the conjugate conduction-natural convection heat transfer along a thin vertical plate with non-uniform internal heat generation, addressing the complex interplay between conduction and convection.

Molla et al. (2006) investigated magnetohydrodynamic natural convection flow on a sphere with uniform heat flux in the presence of heat generation, contributing to the understanding of heat transfer around solid objects. Molla et al. (2004) studied natural convection flow along a vertical wavy surface with uniform surface temperature in the presence of heat generation/absorption, exploring the impact of surface geometry.

Parveen and Alim (2011) focused on Joule heating effects on magnetohydrodynamic natural convection flow along a vertical wavy surface with viscosity dependent on temperature, considering the interaction between Joule heating and variable viscosity. Poots (1961) investigated laminar natural convection flow in magnetohydrodynamics, addressing fundamental aspects of MHD convection.

Prabhakar and Prabhakararao (2013) explored the effect of power-law temperature variation on a vertical conical annular porous medium, providing insights into heat transfer in porous media. Shariful (2004) conducted research on thermal-diffusion and diffusion thermo effects on magneto-hydrodynamics heat and mass transfer, contributing to the understanding of heat and mass transfer phenomena in magnetohydrodynamics.

Sparrow and Cess (1961a) examined the effect of a magnetic field on free convection heat transfer, shedding light on the influence of magnetic fields on convective heat transfer. Tashtoush and Al-Odat (2004) investigated the magnetic field effect on heat and fluid flow over a wavy surface with a variable heat flux, addressing the impact of magnetic fields in complex flow geometries.

The equations governing the flow are solved numerically applying the shooting method together with Keller box Scheme along with Newton's linearization approximation. We have studied the effects of the Prandtl's number *Pr*, the viscous dissipation parameter *Vd*, magnetic parameter *M*, Joule heating parameter *J*, Heat generation parameter *Q* on the velocity and temperature fields as well as on the skin friction and local heat transfer coefficient.

II. Formulation of the Problem

We consider laminar free convection flow along a vertical plate placed in a calm environment with u and v denoting respectively the velocity components in the x and y direction, where x is vertically upwards and y

is the coordinate perpendicular to x.

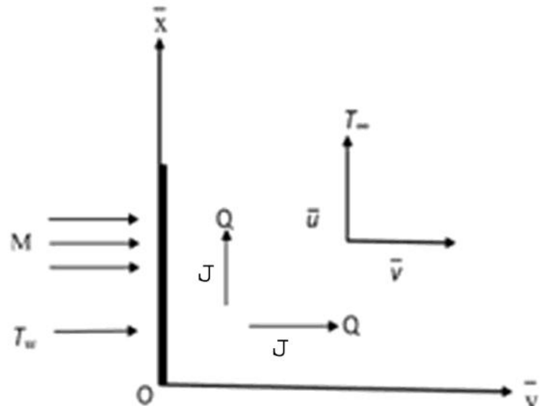


Fig-1

For steady, two-dimensional flow of the boundary layer continuity equation, momentum equation and energy equation including viscous dissipation, heat generation term and Joule heating term are given below.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial v}=\nu\frac{\partial^2 u}{\partial y^2}+g\beta(T-T_\infty)-\frac{\sigma\beta_0^2 u}{\rho} \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho C_p}\frac{\partial^2 T}{\partial y^2}+\frac{\nu}{C_p}\left(\frac{\partial u}{\partial y}\right)^2+\frac{Q_0}{\rho C_p}\left(T-T_\infty\right)+\frac{\sigma\beta_0^2 u^2}{\rho} \tag{3}$$

Where $\alpha=\dfrac{k}{\rho C_p}$ is the thermal diffusivity.

Where x is taken in the direction of the flow i.e. vertically up from the active leading edge for heated upward flows and vertically down for cooled downward flows. The temperature of quiescent ambient fluid T_∞ at large values of y is taken to be constant. Where T_w is the temperature at the wall, T is the fluid temperature, ν is the kinematics velocity, β is the fluid thermal expansion coefficient, β_0 is the magnetic field strength, Q_0 is the heat generation coefficient, C_p is the specific heat at constant pressure, ρ is the fluid density and p is the pressure. The last three terms in the energy equation are the viscous dissipation, heat generation term and the joule heating term respectively.

Equations (1)-(3) are to solved subject to the boundary conditions

$$\begin{aligned} &u=v=o, T=T_w \text{ on } y=0\\ &u\to U, T\to T_\infty \text{ as } y\to\infty\\ &u=U, T=T_w \text{ at } x=0\end{aligned} \tag{4}$$

Where U is the free stream velocity.

The following generalizations are introduced to obtain the equations in terms of generalized stream and temperature functions $f(x,y)$ and $\phi(x,y)$. Now letting

$u(x,y) = \nu\psi_y$

$\Rightarrow u = \nu \frac{\partial \psi}{\partial y}$

$d(x) = \Delta T = T_w - T_\infty$

$v(x,y) = -\nu\psi_x$

$\varphi(x,y) = \frac{T - T_\infty}{\Delta T}$

$i.e. T = T_\infty + \varphi \Delta T$

$\frac{\partial T}{\partial x} = \Delta T \frac{\partial \phi}{\partial x}$ Where, $T_\infty as$ a Constant

Now

$u = \nu \frac{\partial \psi}{\partial y}$

$\therefore \frac{\partial u}{\partial x} = \nu \frac{\partial^2 \psi}{\partial x \partial y}, \frac{\partial u}{\partial y} = \nu \frac{\partial^2 \psi}{\partial y^2}, \frac{\partial^2 u}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3}$

Equation (2) becomes

$$\Rightarrow \psi_y \psi_{yx} - \psi_x \psi_{yy} = \psi_{yyy} + \frac{g\beta\phi\Delta T}{\nu^2} - \frac{\sigma\beta_0^2}{\rho\nu}\psi_y^2 \quad (5)$$

Again, we know

$\frac{\partial T}{\partial x} = \frac{\partial}{\partial x}(T_\infty + \phi\Delta T) = \frac{\partial T_\infty}{\partial x} + \frac{\partial}{\partial x}(\phi\Delta T)$

$\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}[T_\infty + \phi\Delta T]$

$= \Delta T \phi_y$

T_∞ as a constant and ΔT is a function of x.

$$\frac{\partial^2 T}{\partial y^2} = \Delta T \phi_{yy}$$

$$\frac{\partial u}{\partial y} = \nu \frac{\partial^2 \psi}{\partial y^2}$$

$$P_r = \frac{\nu}{\alpha} = \frac{\mu c_p}{k}$$

$$R_e = \frac{\nu l_c}{\nu} = \frac{\rho \nu l_c}{\mu}$$

$$\therefore \frac{1}{\nu} = \frac{\rho}{\mu}$$

$$\nu = \alpha \frac{\mu C_p}{k}$$

$$\frac{\mu}{\rho} = \alpha \frac{\mu C_p}{k}$$

$$\therefore \frac{1}{\rho} = \frac{\alpha}{k} C_p$$

Now equation (3) becomes

$$\Rightarrow \psi_y \phi_x \Delta T + \phi \psi_y (\Delta T)_x - \psi_x \phi_y \Delta T = \frac{k}{\nu \rho c_p} \Delta T \phi_{yy} + \frac{\nu^2}{c_p} \psi_{yy}{}^2 + \frac{Q_0}{\nu \rho C_p} \Delta T + \frac{\sigma B_0^2}{\rho} \nu \psi_y{}^2 \quad (6)$$

Where $(T_0 - T_\infty)$ is the downstream temperature difference (along the x-axis) which would result without the inclusion of the viscous dissipation and joule heating term that are both $\varepsilon(x)$ and $J(x)$ are zero. Gr_x is related to the actual physical Grashof number $Gr_x = Gr_x \phi(0)$

III. Transformation of the Governing Equations

Assuming a similarity variable and a stream function of the following form:

$$x = \xi, \eta = yb(x), \psi(x,y) = c(\xi) f(\eta), \phi(x,y) = \phi(\eta)$$

$$\frac{\partial \psi}{\partial y} = c(\xi) f'(\eta) \frac{\partial}{\partial y}(yb(\xi))$$

$$\therefore \psi_y = c(\xi) f'(\eta) b(\xi)$$

$$\psi_{yy} = c(\xi) b^2(\xi) f''(\eta)$$

$$(\psi_{yy})^2 = c^2(\xi) b^4(y) f''^2(\eta)$$

$$\psi_{yyy} = c(\xi) b^3(\xi) f'''(\eta)$$

$$(\psi_{yy})^2 = c^2(\xi) b^4(y) f''^2(\eta)$$

$$\psi_{xy} = \frac{\partial}{\partial x}\left[c(\xi)b(\xi) f'(\eta)\right]$$
$$= \frac{\partial}{\partial \xi}\left[c(\xi)b(\xi) f'(\eta)\right]$$
$$= c_{\xi}(\xi)b(\xi) f'(\eta) + c(\xi)b_{\xi}(\xi) f'(\eta) + \eta c(\xi)b_{\xi}(\xi) f''(\eta)$$
$$\psi_{x} = \frac{\partial}{\partial x}\left[c(\xi) f(\eta)\right]$$
$$= c_{\xi}(\xi) f(\eta) + c(\xi) f'(\eta)\frac{\eta}{b(\xi)} b_{\xi}(\xi)$$

So, viscous dissipation terms can be written as

$$\frac{\nu^2}{c_p}\psi_{yy}^2 = \frac{\nu^2}{c_p} c^2(\xi)b^4(\xi) f''^2(\eta)$$

and also, the joule heating term

$$\frac{\sigma\beta_0^2}{\rho}\nu\psi_y{}^2 = \frac{\sigma\beta_0^2}{\rho}\nu c^2(\xi)b^2(\xi) f'^2(\eta)$$

Temperature profiles

$$\varphi_y = \varphi'(\eta)b(\xi)$$
$$\varphi_{yy} = \varphi''(\eta)b^2(\xi)$$
$$\varphi_x = \frac{\eta}{b(\xi)} b_{\xi}(\xi).\varphi'(\eta)$$

Therefore, the momentum equation

$$\Rightarrow \psi_y\psi_{yx} - \psi_x\psi_{yy} = \psi_{yyy} + \frac{g\beta\phi\Delta T}{\nu^2} - \frac{\sigma\beta_0^2}{\nu^2}\psi_y$$

$$\Rightarrow \frac{f'^2(\eta) - f(\eta) f''(\eta)}{b(\xi)} c_{\xi}(\xi) + \frac{c(\xi)b_{\xi}(\xi)}{b^2(\xi)} f'^2(\eta) = f'''(\eta) + \frac{g\beta\varphi\Delta T}{c(\xi)b^3(\xi)\nu^2} - \frac{\sigma\beta_0^2}{\mu b^2} f'$$

$$\Rightarrow f'''(\eta) + \frac{g\beta\phi\Delta T}{c(\xi)b^3(\xi)\nu^2} + \frac{c_{\xi}(\xi)}{b(\xi)} f(\eta) f''(\eta) - \left[\frac{c_{\xi}(\xi)}{b(\xi)} + \frac{c(\xi)b_{\xi}(\xi)}{b^2(\xi)}\right] f'^2(\eta) + - \frac{\sigma\beta_0^2}{\mu b^2} f' = 0 \quad (7)$$

Also, the energy equation

$$\psi_y\phi_x\Delta T + \phi\psi_y(\Delta T)_x - \psi_x\phi_y\Delta T = \frac{1}{p_r}\Delta T\phi_{yy} + \frac{\nu^2}{c_p}\psi_{yy}^2 + \frac{Q_o}{\nu\rho C_p}\phi\Delta T + \frac{\sigma\beta_0^2}{\rho}\nu\psi_y{}^2$$

$$\Rightarrow c(\xi)\frac{b_{\xi}(\xi)}{b^2(\xi)} f'(\eta)\varphi'(\eta) + \varphi c(\xi)\frac{f'(\eta)(\Delta T)_{\xi}}{b(\xi)\Delta T} - \frac{c_{\xi}(\xi) f(\eta)\varphi'(\eta)}{b(\xi)} - \eta\frac{c(\xi)b_{\xi}(\xi)}{b^2(\xi)} f(\eta)\varphi(\eta)$$

$$= \frac{1}{p_r}\phi''(\eta) + \frac{\nu^2}{c_p} c^2(\xi)\frac{b^2(\xi)}{\Delta T} f(\eta) + \frac{Q_o}{b^2(\xi)\nu\rho C_p}\phi + \frac{\sigma\beta_0^2}{\rho\Delta T}\nu c^2(\xi) f'^2(\eta)$$

Divide $\Delta T b^2(\xi)$

$$\Rightarrow \frac{1}{p_r}\phi''(\eta)+\frac{c_\xi(\xi)}{b(\xi)}f(n)\phi'(\eta)-\frac{c(\xi)(\Delta T)_\xi}{b(\xi)\Delta T}f'(\eta)\phi(n)+$$

$$\frac{\nu^2}{c_p\Delta T}c^2(\xi)b^2(\xi)f''^2(\eta)-\frac{Q_o}{b^2(\xi)\nu\rho C_p}\phi+\frac{\sigma\beta_0^2}{\rho\Delta T}\nu c^2(\xi)f'^2(\eta)=0 \tag{8}$$

The last three terms in the energy equation (8) are the viscous dissipation, heat generation term and joule heating effect respectively. It is well known that when these terms are neglected, similarity solutions exist for a power law $\Delta T = Nx^n = N\xi^n$ surface (at η =0) temperature distributions.

$$c(x)=4\left(\frac{1}{4}Gr_x\right)^{\frac{1}{4}}, b(x)=\frac{1}{x}\left(\frac{1}{4}Gr_x\right)^{1/4}, Gr_x=\frac{g\beta x^3 n x^n}{\nu^2}$$

$$\therefore c_x=\frac{d}{dx}[4(\frac{1}{4}Gr_x)^{1/4}]$$

$$=4\left(\frac{1}{4}\frac{g\beta n}{\nu^2}\right)^{1/4}\frac{n+3}{4}x^{\frac{n+3}{4}-1}$$

$$=\frac{n+3}{x}\left(\frac{1}{4}\frac{g\beta n}{\nu^2}x^n x^3\right)^{1/4}=\frac{n+3}{x}\left(\frac{1}{4}Gr_x\right)^{1/4}$$

$$b_x=\frac{d}{dx}\left[\frac{1}{x}[\frac{1}{4}\frac{g\beta x^3 n x^n}{\nu^2}]^{1/4}\right]$$

$$=\left[\frac{1}{4}\frac{g\beta n}{\nu^2}\right]^{1/4}\frac{n-1}{4}x^{\frac{n+3}{4}-2}=\frac{n-1}{4x^2}(\frac{1}{4}Gr_x)^{1/4}$$

Now

$$(\Delta T)_x=\frac{d}{dx}(Nx^{\eta)}=nNx^{\eta-1}=\frac{n}{x}\Delta T$$

The for the momentum equation

$$\Rightarrow f'''(\eta)+\frac{g\beta\phi\Delta T}{c(\xi)b^3(\xi)\nu^2}+\frac{c_\xi(\xi)}{b(\xi)}f(\eta)f''(\eta)-\left[\frac{c_\xi(\xi)}{b(\xi)}+\frac{c(\xi)b_\xi(\xi)}{b^2(\xi)}\right]f'^2(\eta)+\frac{\sigma\beta_0^2}{\mu b^2}f'=0 \tag{9}$$

$$\Rightarrow f'''(\eta)+\frac{g\beta\phi N.\xi^{n+3}}{Gr_\xi\nu^2}+(n+3)f(\eta)f''(\eta)-[n+3+n-1]f'^2(\eta)+\frac{\sigma\beta_0^2}{\mu}\frac{2x^2}{\left[\frac{g\beta x^3 N x^n}{\nu^2}\right]^{1/2}}f'=0$$

$$\Rightarrow f'''(\eta)+(n+3)f(\eta)f''(\eta)-2(n+1)f'^2(\eta)+\phi(\eta)+H_a f'=0 \tag{10}$$

Where, $H_a=\frac{2\sigma\beta_0^2\xi^2}{\mu\left(Gr_\xi\right)^{\frac{1}{2}}}$, Hartman number related to MHD.

And energy equation

$$\Rightarrow \frac{1}{p_r}\phi''(\eta)+(n+3)f(\eta)\phi'(\eta)-4n\,f'(\eta)\phi(\eta)+\frac{4\nu^2}{C_p\Delta T}\frac{Gr_\xi}{\xi^2}f''^2(\eta)-\frac{Q_o}{b^2(\xi)\nu\rho C_p}\phi$$
$$+\frac{16\sigma\beta_0^2}{\rho\Delta T}\nu\left(\frac{1}{4}Gr_\xi\right)^{\frac{1}{2}}f'^2(\eta) \qquad (11)$$

Therefore, when viscous dissipation and Joule heating term are included, $f(\eta)$ and $\phi(\eta)$ are functions of η, p_r and ξ^n for the power law case. To retain both the viscous dissipation and Joule heating terms to the first order, $\varepsilon(\xi)$ and $J(\xi)$ are chosen as

$$\varepsilon(\xi)=\frac{4g\beta\xi}{C_p} \text{ and } J(\xi)=\frac{16\sigma\beta_0^2}{\rho\Delta T}\nu\left(\frac{1}{4}Gr_\xi\right)^{\frac{1}{2}}$$

So the equation

$$\phi''(\eta)+p_r[(n+3)f(\eta)\phi'(\eta)-4n\,f'(\eta)\phi(\eta)+\varepsilon(\xi)f''^2(\eta)-Q\phi(\eta)+J(\xi)f'^2(\eta)]=0 \qquad (12)$$

Where, $Q=\dfrac{Q_o}{b^2(\xi)\nu\rho C_p}=\dfrac{Q_0}{\frac{1}{\xi^2}(\frac{1}{4}Gr_\xi)^{\frac{1}{2}}\nu\rho C_p}=\dfrac{Q_0\xi^2}{\nu\rho C_p(\frac{1}{4}Gr_\xi)^{\frac{1}{2}}}$

Q is the dimensionless heat generation parameter.
Equations (3.10) and (3.12) are numerically integrated for the vertical surface case, with the boundary conditions

$$\eta=0:\quad f(0)=0,\quad f'(0)=0,\quad \phi(0)=1; \qquad (13)$$
$$\eta\to\infty:\quad f'(\infty)\to 0,\quad \phi(\infty)\to 0,$$

Skin friction coefficient and heat transfer coefficient:
For the flat surface the heat flux is given by,

$$q''(x)=-k\frac{\partial T}{\partial y}\Big|_{y=0}$$

We know, $T=T_\infty+\phi\Delta T$

$$\therefore \frac{\partial T}{\partial y}=\Delta T\,\phi'(\eta)\frac{\partial\eta}{\partial y}$$
$$=b\Delta T\phi'(\eta)$$

$$\frac{\partial T}{\partial y}\Big|_{y=0}=b\Delta T\phi'(0)$$

The proper $b(x)$ and $\Delta T=d(x)$ give the following heat flux for the power law case.

$$q''(x)=-k\,\Delta T\phi'(0)b(\xi)$$

Where,

$$\Delta T=d(x)=Nx^n$$
$$b(x)=\frac{1}{x}[\frac{1}{4}Gr_x]^{1/4}$$
$$Gr_x=\frac{g\beta x^3N\,x^n}{\nu^2}$$

Heat transfer coefficient (Nusselt Number)

$$Nu_x = \frac{q''(x)}{(T_0 - T_\infty)} \frac{x}{k} = \frac{h_x x}{k}$$

Local Nusselt Number $Nu_\xi = k(T_0 - T_\infty) b(\xi)[-\phi'(0)] \frac{\xi}{k(T_0 - T_\infty)}$

$$= -(\frac{1}{4} Gr_\xi)^{\frac{1}{4}} \phi'(0)$$

The viscous stress $\tau(\xi)$ and Skin friction coefficient C_{f_ξ} defined on a convection velocity

$U = U_c = \nu c(\xi) b(\xi) f'_{\max} \propto \nu c(\xi) b(\xi)$ are know from $f(\eta)$

$$\therefore \tau(\xi) = \mu \frac{\partial u}{\partial y} = \mu \nu \psi_{yy} = \mu \nu c(\xi) f''(\eta) \{b(\xi)\}^2$$

$$\therefore \tau(\xi) = \mu \left. \frac{\partial u}{\partial y} \right|_{y=0} = \mu \nu c(\xi) b^2(\xi) f''(0)$$

When $y = 0, \eta = 0$

Therefore, Skin friction coefficient

$$C_{f\xi} = \frac{\tau(\xi)}{\frac{1}{2}\rho U_c^2} = \frac{2\mu \nu c(\xi) b^2(\xi) f''(0)}{\rho \nu^2 c^2(\xi) b^2(\xi) f'^2_{\max}(0)} \propto \frac{2\mu \nu c(\xi) b^2(\xi) f''(0)}{\rho \nu^2 c^2(\xi) b^2(\xi)}$$

$$= \frac{2 f''(0)}{c(\xi)}$$

$$= \frac{2 f''(0)}{4(\frac{1}{4} Gr_\xi)^{\frac{1}{4}}} = \frac{f''(0)}{2(\frac{1}{4} Gr_\xi)^{\frac{1}{4}}}$$

IV. Results and Discussions

The objective of the present work is to analysis the combined Effects of heat generation and Joule heating on MHD Natural Convection Flow along a Vertical Flat Plate with viscous dissipation and Non-Uniform Surface Temperature.

In this simulation the values of heat generation parameter Q = 1.50, 1.20, 0.90, 0.50, magnetic parameter M = 0.50, 0.40, 0.30, 0.20, 0.10, joule heating parameter J = 0.50, 0.40, 0.30, 0.20, 0.10, Prandlt's number Pr = 7.00, 3.00, 1.00, 0.70 and viscous dissipation Vd = 1.50, 1.30, 1.10, 0.90. If we know the functions $f(\xi, \eta)$, $\phi(\xi, \eta)$ and their derivatives for different values of the heat generation parameter Q, magnetic parameter M, Prandtl's number Pr, viscous dissipation parameter Vd, and the Joule heating parameter J. The velocity and the temperature profiles obtained from the solution are depicted in figure 2 to 6. Also, the local skin friction $f''(\xi, 0)$ and local heat transfer $\phi'(\xi, 0)$ obtained from figure 7 to 11.

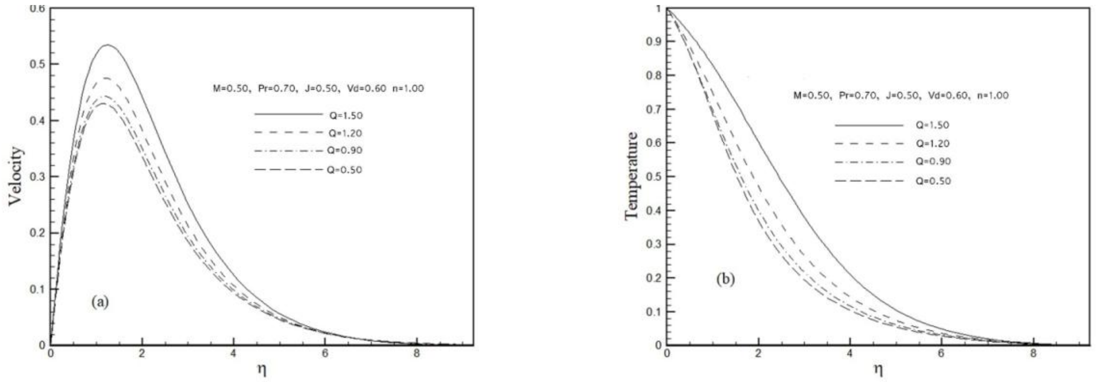


Fig 2a and 2b are displayed for the velocity profiles and temperature profiles for different values of heat generation parameter *Q* = 1.50, 1.20, 0.90, 0.50 against η with other fixed controlling parameter *M* = 0.50, *Pr* = 0.70, *J* = 0.50, *Vd* = 0.60, *n* = 1.00.

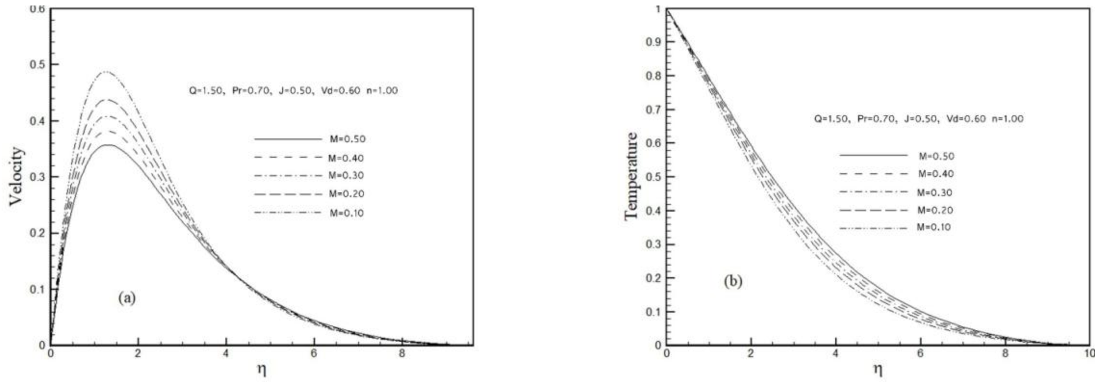


Figure 3a and figure 3b are shown results for the velocity and temperature profiles, for different values of magnetic parameter or Hartmann Number *M* = 0.50, 0.40, 0.30, 0.20, 0.10 plotted against η with fixed controlling parameter *Q* = 1.50, *Pr* = 0.70, *J* = 0.50, *Vd* = 0.60, and *n* = 1.00.

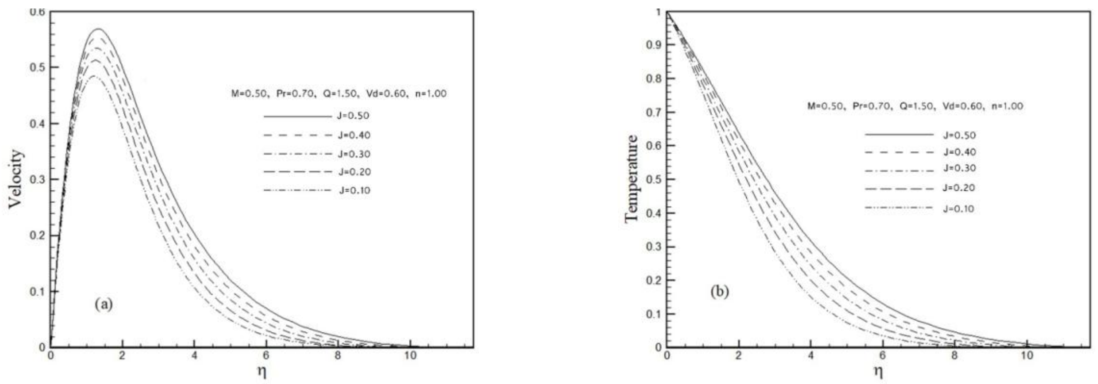


Figure 4a and figure 4b display results for the velocity and temperature profiles for different small values of Joule heating parameter *J* = 0.50, 0.40, 0.30, 0.20, 0.10 plotted against η at *M* = 0.50, *Pr*=0.70, *Q* = 1.50, *Vd* = 0.60, and *n* = 1.00.

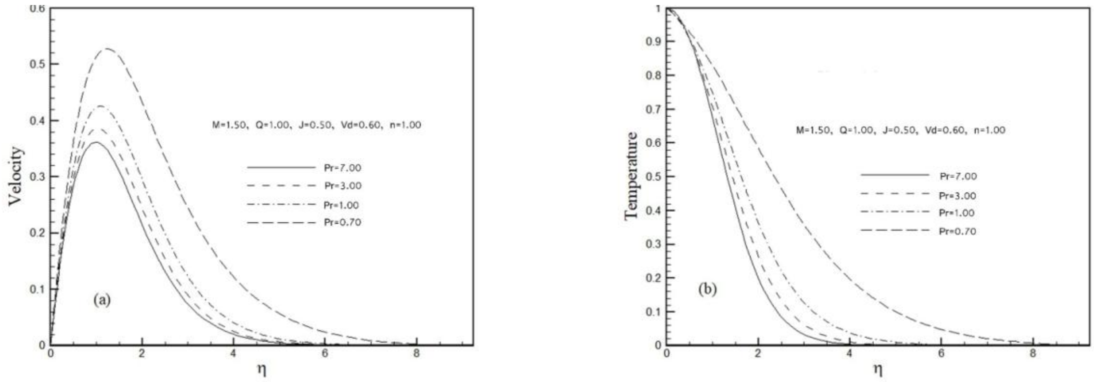


Figure 5a and 5b depicts the velocity profiles and temperature profiles for different Prandtl's number *Pr* = 7.00, 3.00, 1.00, 0.70 plotted against η fixed controlling parameter *M* = 1.50, *Q* = 1.00, *J* = 0.50, *Vd* = 0.60, n = 1.00.

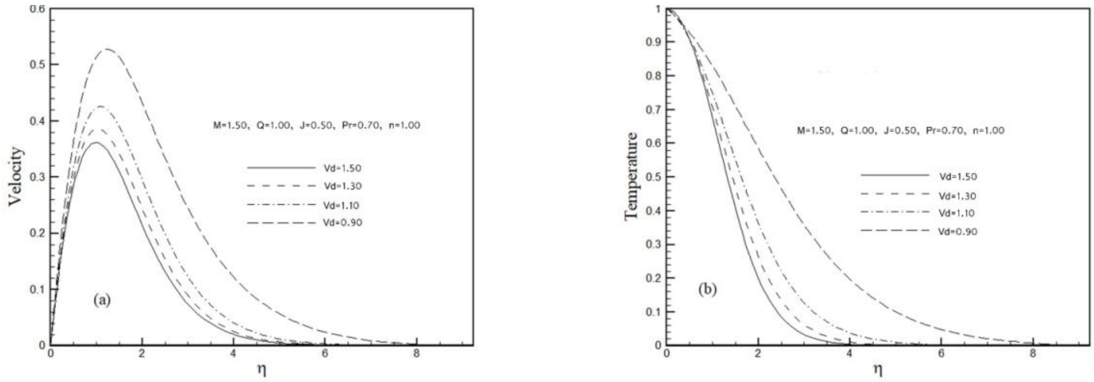


Figure 6a and 6b depicts the velocity profiles and temperature profiles for different viscous dissipation parameter *Vd* = 1.50, 1.30, 1.10, 0.90 plotted against η fixed controlling parameter *M* = 1.50, *Q* = 1.00, *J* = 0.50, *Pr* = 0.70, *n* = 1.00.

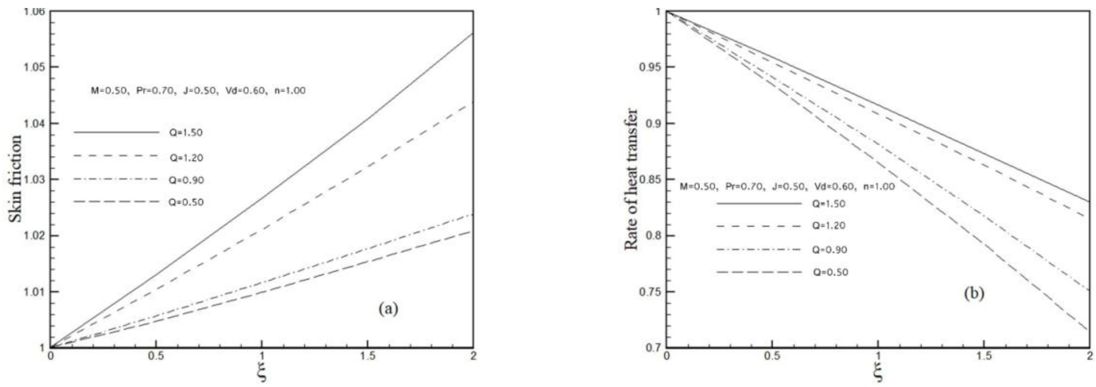


Figure 7a and 7b display results for the skin friction coefficients and local heat transfer coefficients are displayed for different values of Heat generation parameter *Q* = 1.50, 1.20, 0.90, 0.50 against ξ with other controlling parameter *M* = 0.50, *Pr* = 0.70, *J* = 0.50, *Vd* = 0.60, *n* = 1.00.

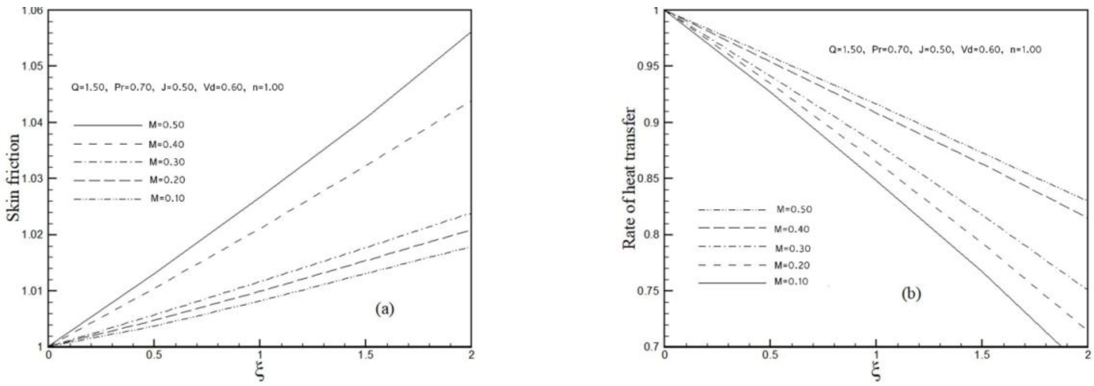


Figure 8a and 8b are displayed skin friction coefficients and local heat transfer coefficients are displayed for different values of magnetic parameter *M* = 0.50, 0.40, 0.30, 0.20, 0.10 against ξ with other controlling parameter *Q* = 1.50, *Pr* = 0.70, *J* = 0.50, *Vd* = 0.60, *n* = 1.00.

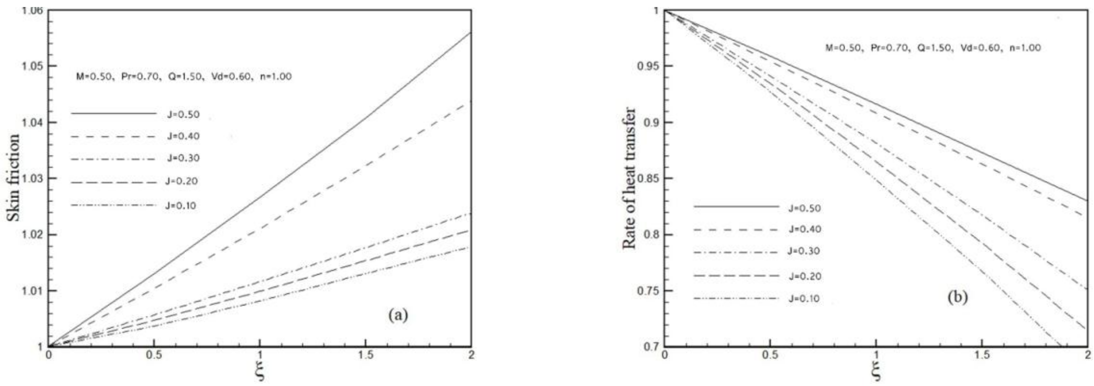


Figure 9a and 9b are shown skin friction coefficients and local heat transfer coefficients are displayed for different values Joule heating parameter *J* = 0.50, 0.40, 0.30, 0.20, 0.10 against ξ with other controlling parameter *M* = 0.50, *Pr* = 0.70, *Q* =1.50, *Vd* = 0.60, *n* = 1.00

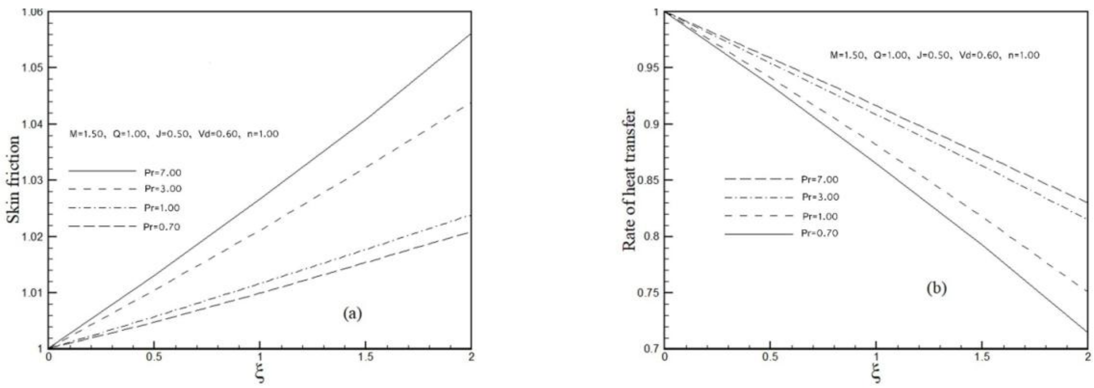
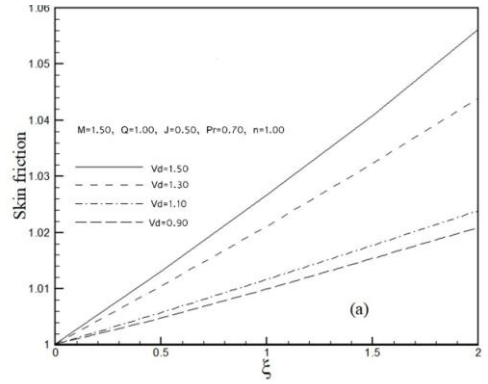


Figure 10a and 10b depicts the skin friction coefficients and local heat transfer coefficients are displayed for different values Prandlt's number *Pr* = 7.00, 3.00, 1.00, 0.70 against ξ with other controlling parameter *M* = 1.50, *Q* = 1.00, *J* = 0.50, *Vd* = 0.60, *n* =1.00.



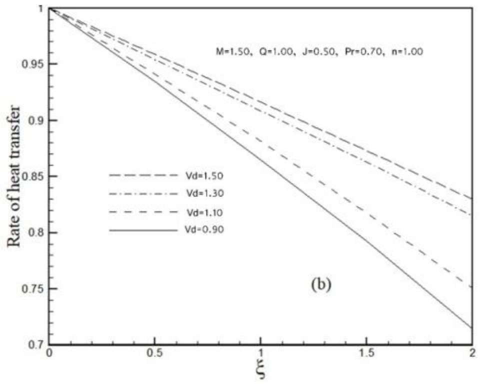


Figure 11a and 11b deal with skin friction coefficients and local heat transfer coefficients are displayed for different values viscous dissipation parameter *Vd* = 1.50, 1.30, 1.10, 0.90, against ξ with other controlling parameter *M* = 1.50, *Q* = 1.00, *J* = 0.50, *Pr* = 0.70, *n* = 1.00.

Fig 2a and 2b display results for the effect of the heat generation parameter *Q* = 1.50, 1.20, 0.90, 0.50 for different values of the controlling parameter *M* = 0.50, *Pr* = 0.70, *J* = 0.50, *Vd* = 0.60, *n* = 1.00 on the velocity profile $f'(\xi, \eta)$ increases with the increase of heat generation parameter *Q* which indicates that the heat generation parameter increases the fluid motion. In figure 2b it is shown that the temperature profile $\phi(\xi, \eta)$ small increase for increasing values of *Q* with others controlling parameter.

The effects of magnetic parameter or Hartmann number *M* = 0.50, 0.40, 0.30, 0.20, 0.10 for *Q* = 1.50, *Pr* = 0.70, *J* = 0.50, *Vd* = 0.60, and *n* = 1.00 on the velocity profiles and temperature profiles are shown in figure 3a and 3b. Figure 3a and figure 3b represent respectively the effects of magnetic parameter *M* on the velocity and temperature profiles. It is seen that the velocity profiles decrease with the increasing value of *M*, but the temperature profiles increase with the increasing value of *M*.

In fig 4a and 4b illustrate the effect of Joule heating parameter *J* = 0.50, 0.40, 0.30, 0.20, 0.10 for *M* = 0.50, *Pr*=0.70, *Q* = 1.50, *Vd* = 0.60, and *n* = 1.00 on the velocity profiles and temperature profiles are shown in figure 4a and 4b. Figure 4a and figure 4b represent respectively the effects of Joule heating parameter *J* on the velocity and temperature profiles. In this figure, it is seen that the velocity profiles increase with the increasing value of *J* and the temperature profiles also increase with the increasing value of Joule heating parameter *J*.

Fig 5a depicts the velocity profiles for different values of Prandtl's number, *Pr* = 7.00, 3.00, 1.00, 0.70 while the other controlling parameter *M* = 1.50, *Q* = 1.00, *J* = 0.50, *Vd* = 0.60, *n* = 1.00. Corresponding distribution of the temperature profiles $\phi(\xi, \eta)$ in the fluids is shown in fig 5b. From fig 5a, it is seen that if the Prandtl's number increase, the velocity of the fluid decrease. On the other hand, from figure 5b it is observed that the temperature profiles decrease within the boundary layer due to increase of the Prandtl's number *Pr*.

The effects of Viscous dissipation parameter *Vd* = 1.50, 1.30, 1.10, 0.90 for *M* = 1.50, *Q* = 1.00, *J* = 0.50, *Pr* = 1.50, *n* = 1.00 on the velocity profiles and temperature profiles are shown in figure 6a and 6b. Figure 6a and figure 6b represent respectively the effects of viscous dissipation parameter *Vd* on the velocity and temperature profiles. The velocity and the temperature profiles decrease with the increasing value of *Vd*.

Numerical values of the skin friction $f''(\xi, 0)$ and the local heat transfer coefficient $\phi'(\xi, 0)$ are depicted graphically in fig 3.7a and 3.7b respectively against ξ for different values of the heat generation parameter *Q* = 1.50, 1.20, 0.90, 0.50 for *M* = 0.50, *Pr* = 0.70, *J* = 0.50, *Vd* = 0.60, *n* = 1.00. It is seen from fig 7a that the skin friction $f''(\xi, 0)$ increases when the heat generation parameter *Q* increases. It is observed in fig 7b, the local heat transfer coefficient $\phi'(\xi, 0)$ increases while the heat generation parameter *Q* increases.

The effects of magnetic parameter or Hartmann number M = 0.50, 0.40, 0.30, 0.20, 0.10 for Q = 1.50, Pr = 0.70, J = 0.50, Vd = 0.60, and n = 1.00 on the local skin friction coefficient $f''(\xi, 0)$ and the local heat transfer coefficient $\phi'(\xi, 0)$ are shown in figure 8a and 8b. Figure 8a and figure 8b it is evident that the increasing value of M leads to increase the skin friction coefficient $f''(\xi, 0)$ and decrease the local heat transfer coefficient $\phi'(\xi, 0)$.

The variation of skin friction and heat transfer for different values of Joule heating parameter J = 0.50, 0.40, 0.30, 0.20, 0.10 against ξ for M = 0.50, Pr=0.70, Q = 1.50, Vd = 0.60, and n = 1.00 are shown in figure 9a and 9b. Figure 9a and figure 9b it is found that the increasing values of Joule heating parameter J leads to increase both the skin friction coefficient $f''(\xi, 0)$ and the local heat transfer coefficient $\phi'(\xi, 0)$.

From fig 10a, it is observed that increase of the value of prandtl's number Pr = 7.00, 3.00, 1.00, 0.70 leads to increase of the value of skin friction Coefficient $f''(\xi, 0)$. But the local heat transfer coefficient $\phi'(\xi, 0)$ decreases shown in figure 10b for the same values of Prandlt's number Pr against ξ M = 1.50, Q = 1.00, J = 0.50, Vd = 0.60, n = 1.00.

Numerical values of the skin friction $f''(\xi, 0)$ and the local heat transfer coefficient $\phi'(\xi, 0)$ are depicted graphically in fig 11a and 11b respectively against ξ for different values of the viscous dissipation parameter Vd = 1.50, 1.30, 1.10, 0.90 for M = 1.50, Q = 1.00, J = 0.50, Pr = 0.70, , n = 1.00. It is seen from fig 11a that the skin friction $f''(\xi, 0)$ increases when the viscous dissipation parameter Vd increases. It is observed in fig 11b, the local heat transfer coefficient $\phi'(\xi, 0)$ decreases while the viscous dissipation parameter Vd increases.

V. Conclusions

From the present investigation, the following conclusions may be drawn:

- Increase in the values of the viscous dissipation parameter Vd leads to decrease the temperature profile, the local heat transfer coefficient and the velocity profile but the local skin friction coefficient increase with the increasing value of Vd.
- Increased values of the Joule heating parameter J lead to increase the velocity profiles. The skin friction coefficient, the temperature profile and the local heat transfer coefficient.
- It has been observed that the temperature profile and the velocity profile decreases for increasing value of Pr but the skin friction coefficient and the local heat transfer coefficient increases with the increasing value of Pr.
- The effect of magnetic parameter or Hartmann Number M is to increase the temperature profiles, the local skin friction coefficient $f''(\xi, 0)$ and the local heat transfer coefficient but the velocity profiles decreases with the increasing value of M.
- Increased values of the heat generation parameter Q leads to increase the velocity profiles, the temperature profiles, the local skin friction coefficient $f''(\xi, 0)$ and the local heat transfer coefficient while M = 0.50, Pr = 0.70, J = 0.50, Vd = 0.60, n = 1.00 .

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